Fluid level in a reservoir with an on-off source

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ABSTRACT

We obtain the Laplace transform of the fluid level probability density function, in terms of the on-period density function, for a fluid queue (or reservoir) with on-off input at equilibrium. We further obtain explicit expressions for the moments of fluid level in terms of the moments of the on-period and hence derive an algorithm for the moments of fluid level at every queue in a tandem network. It turns out that to calculate the kth moment at the ith queue, only the first $k+1$ moments of the on-period of the input process to the first queue are required.

1. INTRODUCTION

Markov modulated fluid queues provide a simple model for shared resources with high intensity input processes; for example routers characterised by bit-rate. They have the advantage of being able to describe correlated input processes, which is often an essential requirement in the modelling of internet traffic and the address traces typical in large-scale storage systems [4, 3]. They are particularly important in a compositional analysis of networks of fluid queues based on a 'building block' queue [2]. The internal flows in the network are approximated by simple (i.e. with small numbers of states) Markov modulated arrival processes (MMAPs). The net input to each node is the superposition of an external fluid flow together with the *output* processes feeding into the node from other nodes in the network. These output processes themselves are approximated by MMAPs and the superposition may be aggregated into a single MMAP under the approximating assumption that the constituent flows are independent. The building block for each node is then defined by the mapping of a single input MMAP into a single output MMAP. This compositional approach has led to accurate approximations in simple networks – remarkably so in tandem networks.

The first contribution of this paper is to find an exact relationship between the Laplace transforms of the probability density functions of the on-period and fluid level random variables in a fluid queue at equilibrium, with constant service rate and exponential off-periods. Since a corresponding relationship between on-period and busy period is known already, it is in principle possible to determine the Laplace transform of the fluid level distribution at any queue in a tandem network, fed by an on-off process at the first node. Such a relationship is complex but we find an explicit algorithm for calculating the moments of fluid level exactly at each queue, finding that one more moment is required of the on-period at each queue (i.e. the busy period of a previous queue) in the tandem; and this in turn requires the same number of moments of the on-period of the source at the first queue.

2. FLUID LEVEL IN A SINGLE RESERVOIR

We now consider a single fluid queue, comprising a server that outputs fluid at a constant rate when it has a positive quantity of fluid stored in its reservoir, or buffer, and a time-homogeneous on-off input (or arrival) stream. Off-periods are exponential random variable and on-periods are general. The fluid input rate is the constant λ during on-periods and zero during off-periods, the fluid output rate from the reservoir is the constant μ when the reservoir is non-empty and the rate of switching from off to on state in the input process is γ, also constant. Thus the net input rate in an on-period is $r^+ = \lambda - \mu$ and the net output rate during an off period is $r^- = \mu$. We define the following random variables for positive integers n:

L_t	fluid level at time t;
T_n	time at which nth on-period begins;
$X_n \equiv L_{T_n}$	fluid level at time T_n;
A_n	increment in fluid level during the nth on-period;
B_n	length of nth on-period;
B	length of a generic on-period;
E_n	length of nth off-period, exponentially distributed with parameter γ.

This fluid queue is well known to have an equilibrium state if and only if $\lambda < \mu(1+\gamma\mathbf{E}[B])$, which condition we assume to hold.

For a generic continuous random variable Z, we denote its probability distribution function by $Z(t) = \mathbf{Pr}[Z \le t]$ and the Laplace-Stieltjes transform (LST) of this distribution by $Z^*(\theta) = \mathbf{E}\big[e^{-\theta Z}\big]$. We denote the density function by $z(t) = Z'(t)$, the derivative of the distribution function, with Laplace transform $Z^*(\theta)$.

2.1 Fluid level at the beginning of on-periods at equilibrium

PROPOSITION 1. *Assuming that the fluid reservoir defined above is at equlibrium, the probability density function of the fluid level at the beginning of an on-period has Laplace transform*

$$X^*(\theta) = \frac{\theta r^-(1-\gamma\mathbf{E}[B]r^+/r^-)}{\theta r^- - \gamma(1-B^*(r^+\theta))}$$

PROOF. We first note that

$$X_{n+1} = \max(0, X_n + r^+B_n - r^-E_n)$$

Therefore

$$\begin{aligned} X_{n+1}^*(\theta) &= \mathbf{E}\Big[e^{-\theta\max(0,X_n+r^+B_n-r^-E_n)}\Big] \\ &= \mathbf{E}\Big[\mathbf{E}\Big[e^{-\theta\max(0,X_n+r^+B_n-r^-E_n)} \mid X_n,B_n\Big]\Big] \\ &= \mathbf{E}\left[\gamma\int_0^{(X_n+r^+B_n)/r^-} e^{-\gamma x+\theta r^- x-\theta(X_n+r^+B_n)}\mathrm{d}x\right] \\ &\quad +\mathbf{E}\left[\gamma\int_{(X_n+r^+B_n)/r^-}^{\infty} e^{-\gamma x}\mathrm{d}x\right] \\ &= \frac{\gamma}{\gamma-\theta r^-}\mathbf{E}\Big[-e^{-(X_n+r^+B_n)\gamma/r^-}+e^{-\theta(X_n+r^+B_n)}\Big] \\ &\quad +\mathbf{E}\Big[e^{-(X_n+r^+B_n)\gamma/r^-}\Big] \\ &= \frac{\gamma}{\gamma-\theta r^-}\mathbf{E}\Big[e^{-\theta(X_n+r^+B_n)}\Big] \\ &\quad -\frac{\theta r^-}{\gamma-\theta r^-}\mathbf{E}\Big[e^{-(X_n+r^+B_n)\gamma/r^-}\Big] \\ &= \frac{\gamma}{\gamma-\theta r^-}X_n^*(\theta)B_n^*(r^+\theta) \\ &\quad -\frac{\theta r^-}{\gamma-\theta r^-}X_n^*(\gamma/r^-)B_n^*(\gamma r^+/r^-) \end{aligned}$$

since X_n and B_n are independent. Thus

$$\begin{aligned}(\gamma-\theta r^-)X_{n+1}^*(\theta) &= \gamma X_n^*(\theta)B_n^*(r^+\theta) \\ &\quad -\theta r^- X_n^*(\gamma/r^-)B_n^*(\gamma r^+/r^-)\end{aligned}$$

Differentiating at $\theta=0$ gives

$$\begin{aligned}\gamma\mathbf{E}[X_{n+1}]+r^- &= \gamma(\mathbf{E}[X_n]+r^+\mathbf{E}[B_n]) \\ &\quad +r^-X_n^*(\gamma/r^-)B_n^*(\gamma r^+/r^-)\end{aligned}$$

Now, assuming the system has an equilibrium state, as $n\to\infty$, let $X_n(t)\to X(t)$ and $X_n^*\to X^*$, so we have, noting all the B_n are i.i.d. as B,

$$(\gamma-\theta r^-)X^*(\theta)=\gamma X^*(\theta)B^*(r^+\theta)-\theta r^-X^*(\gamma/r^-)B^*(\gamma r^+/r^-)$$

and

$$X^*(\gamma/r^-)B^*(\gamma r^+/r^-)=1-\gamma\mathbf{E}[B]r^+/r^-$$

We therefore obtain the required result.

Note that X is equal in distribution to the stationary workload of an $M/G/1$ queue with arrival rate $\frac{\gamma}{r^-}$ and job sizes equal in distribution to r^+B. Using this observation, we can obtain explicit solutions for a much larger class of arrival processes than considered in this paper. □

2.2 Steady state probabilities

The stochastic process describing the fluid reservoir under consideration is regenerative – the beginnings or ends of on-periods, with any given fluid level, are regeneration points – and the end of an off-period occurs as a Poisson process (conditional on being in an off-period) with rate γ. The "Conditional PASTA" result of [5] therefore holds and we have

$$\lim_{t\to\infty}\mathbf{Pr}[L_t\le x\mid S_t=\text{off}]=\lim_{n\to\infty}\mathbf{Pr}\Big[L_{T_n^-}\le x\Big]$$

where $S_t\in\{\text{on},\text{off}\}$ is the state of the arrival process at time t. By Proposition 1, this has LST $X^*(\theta)$.

Now suppose that in the interval $[0,t]$, the source state is off/on for total time $t_{\text{off}}/t_{\text{on}}$ respectively, and that the fluid level is less than or equal to x for total time t_1 during off-periods and for total time t_2 during on-periods. Then by the regenerative property, writing $L(x)=\lim_{t\to\infty}\mathbf{Pr}[L_t\le x]$, with LST $L^*(\theta)$,

$$\begin{aligned}L(x) &= \lim_{t\to\infty}\frac{t_1+t_2}{t}=\lim_{t\to\infty}\frac{t_1}{t_{\text{off}}}\frac{t_{\text{off}}}{t}+\lim_{t\to\infty}\frac{t_2}{t_{\text{on}}}\frac{t_{\text{on}}}{t} \\ &= \frac{L(x|\text{off})}{1+\gamma\mathbf{E}[B]}+\frac{\gamma\mathbf{E}[B]L(x|\text{on})}{1+\gamma\mathbf{E}[B]}\end{aligned}$$

where $L(x|\text{off})=\lim_{t\to\infty}\mathbf{Pr}[L_t\le x\mid S_t=\text{off}], L(x|\text{on})=\lim_{t\to\infty}\mathbf{Pr}[L_t\le x\mid S_t=\text{on}]$, with LSTs $L^*(\theta|\text{off}), L^*(\theta|\text{on})$.

The fluid level at an arbitrary, random time U after the start of an on-period is the fluid level at the beginning of the on-period plus r^+U. Therefore, asymptotically,

$$\begin{aligned}L^*(\theta|\text{on}) &= X^*(\theta)\mathbf{E}\Big[e^{-\theta r^+U}\Big]=X^*(\theta)U^*(r^+\theta) \\ &= \frac{X^*(\theta)(1-B^*(r^+\theta))}{r^+\mathbf{E}[B]\theta}\end{aligned}$$

by the backwards recurrence time (or age) formula of renewal theory.

This finally yields

PROPOSITION 2.

$$L^*(\theta)=\frac{(r^+\theta+\gamma(1-B^*(r^+\theta)))(r^--\gamma\mathbf{E}[B]r^+)}{r^+(\theta r^--\gamma(1-B^*(r^+\theta)))(1+\gamma\mathbf{E}[B])}$$

PROOF.

$$\begin{aligned}L^*(\theta) &= \frac{X^*(\theta)}{1+\gamma\mathbf{E}[B]}+\frac{\gamma\mathbf{E}[B]X^*(\theta)(1-B^*(r^+\theta))}{(1+\gamma\mathbf{E}[B])r^+\mathbf{E}[B]\theta} \\ &= \frac{X^*(\theta)(r^+\theta+\gamma(1-B^*(r^+\theta)))}{(1+\gamma\mathbf{E}[B])r^+\theta}\end{aligned}$$

□

3. MOMENTS

3.1 Moments of ON periods

We now consider the following problem: Given n fluid queues in tandem with an on-off source of the kind described in Section 2 at the input queue (queue 1), we wish to obtain the kth moment of the fluid level at every queue. From Proposition 2, this moment can be obtained for the input queue, and further it requires only the first $k+1$ moments of the ON period duration. The question therefore is, how do we extend this analysis to subsequent queues in the tandem network?

The input to queue 2 is the output of queue 1 and is again an on-off source, but the ON periods now are the *busy periods of queue 1*. The OFF periods of queue 2 are also distributed i.i.d. as Exp(γ). To determine the kth moment of the fluid level at queue 2, we therefore have to find the first $k+1$ moments of the busy periods of queue 1. We show below (Section 3.2) that the first k moments of the busy period of such a fluid queue are completely determined by the first k moments of the ON periods of its source. By successively applying this result to each queue in the tandem network, we can see that *the kth moment of the fluid level of each queue is completely determined by the first $k+1$ moments of the durations of the ON periods of the source at queue 1.*

3.2 Busy period of a fluid queue with on-off source

The busy period of a single, constant rate fluid queue with on-off source having exponential on-periods is well understood, see for example [2, 1].

PROPOSITION 3. *Consider an on-off fluid process with exponentially distributed OFF period, parameter γ, constant fluid input rate λ in the ON periods and constant fluid output rate μ when the fluid level is positive. Let the ON period (respectively, busy-period) random variable be denoted by B (respectively, W), with Laplace transform of probability distribution $B^*(\theta)$ (respectively, $W^*(\theta)$). Then*

$$W^*(\theta) = B^*\left(\theta\eta + \gamma(\eta - 1)(1 - W^*(\theta))\right)$$

where $\eta = \lambda/\mu > 1$.

A more general result of [1] determines the Laplace transform of the busy period density when there are multiple MMOAP arrival streams, each of which has on-rate greater than the service rate of the node. This result could be needed to extend our approach to multiple on-off sources. However, for the present study of tandem and treelike networks, the above proposition suffices.

Analogously to the $M/G/1$ queue, there are no *explicit* expressions for the Laplace transform of the busy period. However, expressions for the moments of the busy period can be derived directly by differentiation at $\theta = 0$. The first three moments w_1, w_2, w_3 of the busy period that we require are now obtained thus, and an algorithm for the kth moment $(k = 1, 2, \ldots)$ can be obtained using Leibnitz's rule:

$$\begin{aligned} w_1 &= \frac{\eta b_1}{1 - \gamma(\eta - 1)b_1} \\ w_2 &= \frac{(\eta + \gamma(\eta - 1)w_1)^2 b_2}{1 - \gamma(\eta - 1)b_1} \\ w_3 &= \frac{(\eta + \gamma(\eta - 1)w_1)^3 b_3 + 3\gamma(\eta - 1)b_2 w_2}{1 - \gamma(\eta - 1)b_1} \end{aligned}$$

where b_1, b_2, b_3 are the first three moments of the *on*-period.

3.3 Moment recurrences for a tandem network of fluid queues

In this section we use the Laplace transform for the fluid level derived in Section 2 to give explicit expressions for the moments of the stationary fluid level distribution, in terms of the moments of the source's on-periods. We thus provide a recursive method to obtain the moments of the stationary fluid level distribution at all the queues in the tandem network in terms of the moments of the on-periods of the source at queue 1.

Let,

$$b_i = \mathbf{E}\Big[B^i\Big] \quad , \quad \rho = \frac{\gamma r^+ b_1}{r^-}$$

$$u_i = \mathbf{E}\Big[(r^+U)^i\Big] \ = \ \frac{(r^+)^i b_{i+1}}{(i+1)b_1}$$

$$x_1 = \mathbf{E}[X] \ = \ \frac{\rho}{1-\rho}u_1$$

$$x_2 = \mathbf{E}\big[X^2\big] \ = \ 2\left(\frac{\rho}{1-\rho}\right)^2 u_1^2 + \frac{\rho}{1-\rho}u_2$$

$$x_3 = \mathbf{E}\big[X^3\big] \ = \ 6\left(\frac{\rho}{1-\rho}\right)^3 u_1^3 + 6\left(\frac{\rho}{1-\rho}\right)^2 u_1 u_2 + \frac{\rho}{1-\rho}u_3$$

The first three moments of the fluid level that we require are now obtained thus, and an algorithm for the kth moment $(k = 1, 2, \ldots)$ can again be obtained using Leibnitz's rule:

$$\begin{aligned} \mathbf{E}[L] &= x_1 + \frac{\gamma b_1}{1+\gamma b_1}u_1 \\ \mathbf{E}\big[L^2\big] &= x_2 + 2\frac{\gamma b_1}{1+\gamma b_1}x_1 u_1 + \left(\frac{\gamma b_1}{1+\gamma b_1}\right)^2 u_2 \\ \mathbf{E}\big[L^3\big] &= x_3 + 3\frac{\gamma b_1}{1+\gamma b_1}x_2 u_1 + 3\left(\frac{\gamma b_1}{1+\gamma b_1}\right)^2 x_1 u_2 \\ &\quad + \left(\frac{\gamma b_1}{1+\gamma b_1}\right)^3 u_3 \end{aligned}$$

In fact, we do not need to iterate in this way to get the *busy period* moments at each queue – we can just use the ON period of the source that feeds the first queue in the tandem. Essentially, this is because there will always be a backlog of fluid at any busy node $n > 1$ whether this queue is fed by the previous queue or by the external source directly, since there are never any upstream hold-ups (the upstream fluid output rates being greater than at queue n). This provides a small simplification for our method.

4. CONCLUSION

The remarkable accuracy of the mean fluid levels obtained in [2] for tandem networks is not surprising in view of the result we have proved that only the first two moments of the source are required; these were known exactly and second moments of on-period were matched exactly throughout the tandem. Whilst the present contribution is restricted to tandem networks only, it provides a further generalisation in that both the on-period of the source and all service times can be general. A further generalisation planned includes multiple input streams at the source. These would allow both correlation in input traffic to be represented and the possibility of priority classes, which are needed in modelling internet traffic and address-trace traffic in large-scale data storage.

5. REFERENCES

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